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Computation of periodic orbits in three–dimensional Lotka–Volterra systems

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This paper deals with an adaptation of the Poincaré–Lindstedt method for the determination of periodic orbits in three–dimensional nonlinear differential systems. We describe here a general symbolic algorithm to implement the method, and apply it to compute periodic solutions in a three–dimensional Lotka–Volterra system modeling a chain food interaction. The sufficient conditions to make secular terms dissapear from the approximate series solution are given in the paper.

Keywords: Lotka–Volterra; Periodic orbits; Poincaré–Lindstedt method; Symbolic computation

1. Introduction

In the study of periodic solutions of ordinary differential equations, the classical method of Poincaré–Lindstedt has always been a most attractive one from the point of view of engineers and other applied mathematicians. The Poincaré–Lindstedt technique is a classical perturbation method used to continue a periodic orbit with respect to a small perturbation parameter, when fixing the amplitude (or the energy) of the system. This method has been used to analyze a wide variety of systems in many branches of science: from galactic ([14], [15]) to atomic models [11], and covering also applications in population biology, ecology and mathematical biology [1]. Nowadays, many researchers ([2], [4], [5], [8], [13] and [17] to cite some examples) make use of this method to study dynamical systems. Buonomo [2] uses the Poincaré expansion theorem to give the periodic solution of the van der Pol equation $\ddot{x} + x = \epsilon(1 - x^2)\dot{x}$ in the form of a series converging for all values of the damping parameter ϵ. For ϵ small the series solution reduces to a perturbation series in powers of ϵ and is obtained from this series, essentially by the analytical continuation method, using a suitable transformation of the parameter. The coefficients in the expansions can be obtained by symbolic computation in exact rational number form up to a very high order, the only limitation being the available computing resources. Hu and Xiong [8] explore the capabilities of the Poincaré–Lindstedt method to study the cubic Duffing equation, and the quintic Duffing equation is also examined with this technique by Ramos [13].

The Poincaré–Lindstedt technique is only to solve nonlinear differential equations depending on a small parameter. In [4], Chen and Cheung overcome this restriction by employing a modification of the Poincaré–Lindstedt technique. This method

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involves a parameter transformation such that a strongly nonlinear system with a large parameter is transformed into a small parameter system. Starting from different points of view, He ([6], [7]) extends the classical Poincaré–Lindstedt method and develop two modified versions of the techniques. In [6], He introduces a new Poincaré–Lindstedt method based on the expansion of a constant, rather than the nonlinear frequency, in powers of the expanding parameter to avoid the ocurrence of secular terms in the perturbation series solution. In [7], a new transformation of the independent variable is introduced. This transformation avoid the occurrence of secular terms in the perturbation series solution. The results show that the obtained approximate solutions are uniformly valid on the whole solution domain, and they are suitable not only for weakly nonlinear systems, but also for strongly nonlinear systems. Xu review the asymptotic techniques developed by He in [18]. In this review, He's parameter–expanding methods including modified Lindstedt–Poincaré method and bookkeeping parameter method are discussed in detail. Some remarkable virtues of the methods are exploited, and their applications are illustrated. A detailed review of some other Poincaré–Lindstedt–type methods can be found in [13].

The literature devoted to the application of the Poincaré–Lindstedt technique to systems of differential equations presenting a periodic orbit is not so abundant. In [16], an approximation to the periodic solutions of the general Lotka–Volterra prey–predator system is obtained using the Poincaré–Lindstedt method. In [10], the method of Poincaré–Lindstedt is adapted to compute periodic solutions in perturbed two–dimensional systems. In particular, an approximate solution to a Lotka–Volterra model for two species is computed. The computation of periodic solutions in Lotka–Volterra systems is an open problem where the Poincaré–Lindstedt method could play a key role in the computation of periodic orbits and the understanding of the way the phase space is structured not only in two species systems.

The aim of this paper is to present a general algorithm for implementing the standard Poincaré–Lindstedt method to three–dimensional perturbed systems of differential equations of first order. This adaptation is successfully applied to compute periodic solutions in Lotka–Volterra systems modeling a three–species food chain interaction. In the following section, we describe how to adapt the standard method to three–dimensional systems of the Lotka–Volterra type and give sufficient conditions to make secular terms dissapear from the solution. This result is key to set the stage to adapt the perturbation method to three–dimensional systems of differential equations.

2. Adaptation of the Poincaré–Lindstedt Method for Three–Dimensional Systems

In [12], Poincaré faces the problem of the determination of periodic solutions by series expansion with respect to a small parameter. Consider for instance the equation $\ddot{x} + x = \epsilon f(x)$, and suppose that an isolated periodic solution exists for $0 < \epsilon \ll 1$, and if $\epsilon = 0$ all solutions are periodic. Under certain conditions Poincaré proves that the periodic solution can be described by a convergent series in entire powers of ϵ, where the coefficients are bounded functions of time. In the next, we give a detailed description of the adaptation of this tecnique to three–dimensional perturbed systems.

Let us consider the problem defined by the following nonlinear differential system of first order,

$$\begin{aligned} \dot{x} + \alpha_{12} y &= \epsilon f_1(x, y, z), \\ \dot{y} - \alpha_{21} x + \alpha_{23} z &= \epsilon f_2(x, y, z), \\ \dot{z} - \alpha_{32} y &= \epsilon f_3(x, y, z), \end{aligned} \tag{1}$$

where $0 < \epsilon \ll 1$ is a small parameter and functions $f_1(x, y, z)$, $f_2(x, y, z)$ and $f_3(x, y, z)$ can be arranged as follows,

$$\begin{aligned} f_1(x,y,z) &= \sum_{0\le q\le M}\sum_{0\le\nu_1\le q}\sum_{0\le\nu_2\le\nu_1} f_{1,\nu_2,q-\nu_1,\nu_1-\nu_2}\, x^{\nu_2} y^{q-\nu_1} z^{\nu_1-\nu_2}\,, \\ f_2(x,y,z) &= \sum_{0\le q\le M}\sum_{0\le\nu_1\le q}\sum_{0\le\nu_2\le\nu_1} f_{2,\nu_2,q-\nu_1,\nu_1-\nu_2}\, x^{\nu_2} y^{q-\nu_1} z^{\nu_1-\nu_2}\,, \\ f_3(x,y,z) &= \sum_{0\le q\le M}\sum_{0\le\nu_1\le q}\sum_{0\le\nu_2\le\nu_1} f_{3,\nu_2,q-\nu_1,\nu_1-\nu_2}\, x^{\nu_2} y^{q-\nu_1} z^{\nu_1-\nu_2}\,, \end{aligned} \tag{2}$$

where $f_{1,\nu_2,q-\nu_1,\nu_1-\nu_2}$, $f_{2,\nu_2,q-\nu_1,\nu_1-\nu_2}$ and $f_{3,\nu_2,q-\nu_1,\nu_1-\nu_2} \in \mathbb{R}$ for $0 \le q \le M$, $0 \le \nu_1 \le q$, $0 \le \nu_2 \le \nu_1$ and $M \in \mathbb{N}$.

If the unperturbed system ($\epsilon = 0$) has periodic solutions and ϵ is a measure of the size of the perturbing terms, then the trajectories for the full system will remain pretty close to those of the non–perturbed system, for any finite period of time $t_0 < t < t_0 + \alpha$ ($\alpha > 0$) with an error not larger than $O(\alpha)$. In general, even a small perturbation is enough to destroy periodicity, that is, nonlinearity will finish with most of the periodic orbits of the unperturbed system, but some of them may persist. The Poincaré–Lindstedt technique is used to find those periodic solutions by expanding the solution of the system in the form

$$\begin{aligned} x(t) &= x_0(T) + \epsilon\, x_1(T) + \epsilon^2\, x_2(T) + \cdots\,, \\ y(t) &= y_0(T) + \epsilon\, y_1(T) + \epsilon^2\, y_2(T) + \cdots\,, \\ z(t) &= z_0(T) + \epsilon\, z_1(T) + \epsilon^2\, z_2(T) + \cdots\,, \end{aligned} \tag{3}$$

where $x_\nu = x_\nu(T)$, $y_\nu = y_\nu(T)$ and $z_\nu = z_\nu(T)$ are 2π–periodic in T, and $T = \omega t$ is the stretched time variable, with

$$\omega = 1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots\,, \tag{4}$$

being ω_ν real constants. Thus, the nonlinear period is $2\pi/\omega$.

To apply this technique, one has to start by rewriting (1) in terms of the new independent variable T, to obtain

$$\begin{aligned} \omega x' + \alpha_{12} y &= \epsilon \sum_{0\le q\le M}\sum_{0\le\nu_1\le q}\sum_{0\le\nu_2\le\nu_1} f_{1,\nu_2,q-\nu_1,\nu_1-\nu_2}\, x^{\nu_2} y^{q-\nu_1} z^{\nu_1-\nu_2}\,, \\ \omega y' - \alpha_{21} x + \alpha_{23} z &= \epsilon \sum_{0\le q\le M}\sum_{0\le\nu_1\le q}\sum_{0\le\nu_2\le\nu_1} f_{2,\nu_2,q-\nu_1,\nu_1-\nu_2}\, x^{\nu_2} y^{q-\nu_1} z^{\nu_1-\nu_2}\,, \\ \omega z' - \alpha_{32} y &= \epsilon \sum_{0\le q\le M}\sum_{0\le\nu_1\le q}\sum_{0\le\nu_2\le\nu_1} f_{3,\nu_2,q-\nu_1,\nu_1-\nu_2}\, x^{\nu_2} y^{q-\nu_1} z^{\nu_1-\nu_2}\,. \end{aligned} \tag{5}$$

Here, $\dot{}$ stands for d/dt and $'$ for d/dT. If expansions (3) and (4) are substituted into (5), and terms in equal powers of ϵ are collected, we get an equation for each order of the approximation in the expansions (3).

Let us introduce introduce here the following notation: S_ν will denote the ν–th order coefficient of the expansion of S, so that

$$S = S_0 + \epsilon S_1 + \epsilon^2 S_2 + \cdots\,.$$

Thus, if $S = x^2$, then $(x^2)_0 = x_0 x_0$, $(x^2)_1 = 2x_0 x_1$, and in general, $(x^2)_q = \sum_{0\le\nu\le q} x_\nu x_{q-\nu}$. This notation simplifies the way to express the formulae for the computation of the coefficients of the expansion of the solution at any order.

The solution to (1) is constructed from the order zero, which corresponds with the unperturbed problem, and can be written as

$$\begin{aligned} \omega_0 x_0' + \alpha_{12} y_0 &= 0\,, \\ \omega_0 y_0' - \alpha_{21} x_0 + \alpha_{23} z_0 &= 0\,, \\ \omega_0 z_0' - \alpha_{32} y_0 &= 0\,. \end{aligned} \tag{6}$$

The first order system is given by

$$\begin{aligned}
\omega_0 x_1' + \alpha_{12} y_1 &= \sum_{0\le q\le M}\sum_{0\le\nu_1\le q}\sum_{0\le\nu_2\le\nu_1} f_{1,\nu_2,q-\nu_1,\nu_1-\nu_2}\, x_0^{\nu_2} y_0^{q-\nu_1} z_0^{\nu_1-\nu_2} - \omega_1 x_0' \,, \\
\omega_0 y_1' - \alpha_{21} x_1 + \alpha_{23} z_1 &= \sum_{0\le q\le M}\sum_{0\le\nu_1\le q}\sum_{0\le\nu_2\le\nu_1} f_{2,\nu_2,q-\nu_1,\nu_1-\nu_2}\, x_0^{\nu_2} y_0^{q-\nu_1} z_0^{\nu_1-\nu_2} - \omega_1 y_0' \,, \\
\omega_0 z_1' - \alpha_{32} y_1 &= \sum_{0\le q\le M}\sum_{0\le\nu_1\le q}\sum_{0\le\nu_2\le\nu_1} f_{3,\nu_2,q-\nu_1,\nu_1-\nu_2}\, x_0^{\nu_2} y_0^{q-\nu_1} z_0^{\nu_1-\nu_2} - \omega_1 z_0' \,.
\end{aligned} \tag{7}$$

The order Q of the expansion is obtained by solving the system

$$\begin{aligned}
\omega_0 x_Q' + \alpha_{12} y_Q &= \sum_{0\le q\le M}\sum_{0\le\nu_1\le q}\sum_{0\le\nu_2\le\nu_1} f_{1,\nu_2,q-\nu_1,\nu_1-\nu_2}\, \left[x^{\nu_2} y^{q-\nu_1} z^{\nu_1-\nu_2}\right]_{Q-1} - \sum_{1\le\nu\le Q-1} x_\nu' \omega_{Q-\nu} - \omega_Q x_0' \,, \\
\omega_0 y_Q' - \alpha_{21} x_Q + \alpha_{23} z_Q &= \sum_{0\le q\le M}\sum_{0\le\nu_1\le q}\sum_{0\le\nu_2\le\nu_1} f_{2,\nu_2,q-\nu_1,\nu_1-\nu_2}\, \left[x^{\nu_2} y^{q-\nu_1} z^{\nu_1-\nu_2}\right]_{Q-1} - \sum_{1\le\nu\le Q-1} y_\nu' \omega_{Q-\nu} - \omega_Q y_0' \,, \\
\omega_0 z_Q' - \alpha_{32} y_Q &= \sum_{0\le q\le M}\sum_{0\le\nu_1\le q}\sum_{0\le\nu_2\le\nu_1} f_{3,\nu_2,q-\nu_1,\nu_1-\nu_2}\, \left[x^{\nu_2} y^{q-\nu_1} z^{\nu_1-\nu_2}\right]_{Q-1} - \sum_{1\le\nu\le Q-1} z_\nu' \omega_{Q-\nu} - \omega_Q z_0' \,.
\end{aligned} \tag{8}$$

At each order p of the perturbation method, one has to calculate x_p, y_p, z_p and ω_p from the equation above, but also x_p', y_p', z_p' and the collection of products $(x^{\nu_1} y^{\nu_2} z^{\nu_3})_p$ for each $\nu_1, \nu_2, \nu_3 \in \mathbb{Z}$ such that $0 \le \nu_1, \nu_2, \nu_3 \le M$, in order to compute the right–hand side of equation (8) for the $(p+1)$–th order of the perturbation method. At the p–th order of the Poincaré–Lindstedt approximation, one first fits the value of ω_p to assure that no secular terms exist, expressing it as a function of some constants which depend on the initial conditions of the problem. This is done by applying the proposition 1, given below. Once ω_p has been obtained, x_p, y_p and z_p can be computed by solving the system (8). In the section 4, we describe a procedure for the implementation of the Poincaré–Lindstedt method.

Proposition 1 *The differential system*

$$\begin{aligned}
\omega_0 x' + \alpha_{12} y &= A\cos T + B\sin T + \sum_{n>1}(A_n \cos nT + B_n \sin nT)\,, \\
\omega_0 y' - \alpha_{21} x + \alpha_{23} z &= C\cos T + D\sin T + \sum_{n>1}(C_n \cos nT + D_n \sin nT)\,, \\
\omega_0 z' - \alpha_{32} y &= E\cos T + F\sin T + \sum_{n>1}(E_n \cos nT + F_n \sin nT)\,,
\end{aligned} \tag{9}$$

where

$$\omega_0 = \sqrt{\alpha_{12}\alpha_{21} + \alpha_{23}\alpha_{32}}\,,$$

has no secular terms if

$$\begin{aligned}
\omega_0 C + \alpha_{23} F - \alpha_{21} B &= 0\,, \\
\omega_0 D - \alpha_{23} E + \alpha_{21} A &= 0\,.
\end{aligned}$$

Proof. Let us take, without loss of generality,

$$A_n = B_n = C_n = D_n = D_n = E_n = F_n = 0\,,$$

for any $n > 1$. The solution $x(t)$ to (9) can be written as

$$\begin{aligned}
x(t) &= \frac{1}{\omega_0^2}\left(\alpha_{12}\omega_0 k_3 - \frac{\alpha_{12}\alpha_{21} + 2\alpha_{23}\alpha_{32}}{2\omega_0} B - \frac{\alpha_{12}}{2} C - \frac{\alpha_{12}\alpha_{23}}{2\omega_0} F\right)\cos T + \\
&+ \frac{1}{\omega_0^2}\left(\frac{\alpha_{12}\alpha_{21}}{2\omega_0} A + \frac{\alpha_{12}}{2} D - \frac{\alpha_{12}\alpha_{23}}{2\omega_0} E\right) T\cos T - \\
&- \frac{1}{\omega_0^2}\left(\alpha_{12}\omega_0 k_2 + \alpha_{12} D - \frac{\alpha_{23}\alpha_{32}}{\omega_0} A - \frac{\alpha_{12}\alpha_{23}}{\omega_0} E\right)\sin T - \\
&- \frac{1}{\omega_0^2}\left(-\frac{\alpha_{12}\alpha_{21}}{2\omega_0} B + \frac{\alpha_{12}}{2} C + \frac{\alpha_{12}\alpha_{23}}{2\omega_0} F\right) T\sin T + k_1\,,
\end{aligned}$$

where k_1, k_2 and k_3 are integration contants depending on the initial conditions of the problem. The condition to make the secular term $T\cos T$ dissapear is given by

$$\frac{\alpha_{12}}{2}D-\frac{\alpha_{12}\alpha_{23}}{2\omega_0}E+\frac{\alpha_{12}\alpha_{21}}{2\omega_0}A=0\,,$$

that is,

$$\omega_0 D-\alpha_{23}E+\alpha_{21}A=0\,.$$

In the same way, the condition to make the secular term $T\sin T$ dissapear is given by

$$\frac{\alpha_{12}}{2}C-\frac{\alpha_{12}\alpha_{21}}{2\omega_0}B+\frac{\alpha_{12}\alpha_{23}}{2\omega_0}F=0\,,$$

that is,

$$\omega_0 C+\alpha_{23}F-\alpha_{21}B=0\,.$$

This procedure can be carried out also for $y(t)$ and $z(t)$, obtaining the same set of conditions. ■

3. Application to Lotka–Volterra Systems

In 1925, Lotka [9] proposed a differential equation model to describe the population dynamics of two interacting species, a predator and its prey. A species $x(t)$ serves as food to another species $y(t)$, so that, in this sense, $x(t)$ becomes transformed into $y(t)$. The formulation of Volterra was as follows: A species $y(t)$ feeds on a species $x(t)$, which, in turn feeds on some source presented in such large excess that the mass of this source may be considered constant during the period of time under consideration. Then, under this assumption, we have that the rate of increase of $x(t)$ per unit of time is equal to the difference between the mass of newly formed of $x(t)$ per unit of time and the mass of $x(t)$ destroyed by $y(t)$ per unit of time. The Lotka–Volterra model for two species consists of the following differential equations:

$$\begin{aligned}\dot{x}&=x(r_1-a_{12}x_2)\,,\\ \dot{y}&=y(-r_2+a_{21}x_2)\,.\end{aligned}\qquad(10)$$

Here, $y(t)$ and $x(t)$ represent, respectively, the predator population and the prey population as functions of time. The parameters $r_1, r_2, a_{12}, a_{21}>0$ are interpreted as follows: r_1 represents the natural growth rate of the prey in the absence of predators, a_{12} represents the effect of predation on the prey, r_2 is the natural death rate of the predator in the absence of prey, and a_{21} the efficiency and propagation rate of the predator in the presence of prey.

In this section, in order to illustrate the effectiveness of the perturbation procedure, we will focus our interest in a linear three species food chain where the lowest level prey, x is preyed upon by a mid–level species, y who, in turn, is preyed upon by a top level predator z. The equations of the system are

$$\begin{aligned}\dot{x}&=x(r_1-a_{12}y)\,,\\ \dot{y}&=y(-r_2+a_{21}x-a_{23}z)\,,\\ \dot{z}&=z(-r_3+a_{32}y)\,.\end{aligned}\qquad(11)$$

Here, $r_1, r_2, r_3>0$ and $a_{12}, a_{21}, a_{23}, a_{32}>0$. The parameters r_1, r_2, a_{12} and a_{21} have the same meaning as in the Lotka–Volterra equations for two species, a_{23} represents the effect of predation on species y by species z, r_3 the natural death rate of the predator z in the absence of prey, and a_{32} stands for the efficiency and propagation rate of the predator

z in the presence of prey. Since populations are non–negative, we will restrict our attention to the non–negative octant $\Omega = \{(x, y, z) : \ x \geq 0\,,\ y \geq 0\,,\ z \geq 0\} \subset \mathbb{R}^3$, and the positive octant $\Omega^+ = \{(x, y, z) : \ x > 0\,,\ y > 0\,,\ z > 0\} \subset \mathbb{R}^3$. Each coordinate plane is invariant with respect to (11) (see for instance [3]).

Let us study now the equilibrium points of (11). This system has equilibrium points in the interior of Ω^+ if

$$\begin{aligned} r_1 - a_{12} y &= 0\,, \\ -r_2 + a_{21} x - a_{23} z &= 0\,, \\ -r_3 + a_{32} y &= 0\,. \end{aligned}$$

By solving these equations, we get that

$$y = \frac{1}{a_{12}} r_1\,, \quad r_3 a_{12} = r_1 a_{32}\,,$$

and

$$x = \frac{1}{a_{21}} r_2 + \frac{a_{23}}{a_{21}} z\,.$$

Thus, it results straightforward to find that there is ray of equilibrium points of the form

$$P_\lambda \left(\frac{r_2}{a_{21}} + \frac{a_{23}}{a_{21}} \lambda, \frac{r_1}{a_{12}}, \lambda \right)\,,$$

with $\lambda \in \mathbb{R}$, in the case $r_3 a_{12} = r_1 a_{32}$. The Jacobian matrix of the system in any of these equilibrium points is given by

$$A(P_\lambda) = \begin{pmatrix} 0 & -a_{12}(r_2 + a_{23}\lambda)/a_{21} & 0 \\ a_{21} r_1/a_{12} & 0 & -a_{23} r_1/a_{12} \\ 0 & a_{32}\lambda & 0 \end{pmatrix}\,,$$

with eigenvalues

$$\begin{aligned} \lambda_1 &= 0\,, \\ \lambda_2 &= +\frac{1}{a_{12}} \sqrt{-a_{12} r_1 \left(a_{23} a_{32} \lambda + r_2 a_{12} + a_{23} a_{12} \lambda \right)}\,, \\ \lambda_3 &= -\frac{1}{a_{12}} \sqrt{-a_{12} r_1 \left(a_{23} a_{32} \lambda + r_2 a_{12} + a_{23} a_{12} \lambda \right)}\,. \end{aligned}$$

As the three eigenvalues have zero real part, each such equilibrium point has a three–dimensional center manifold, which does not help us determine the dynamics near these fixed points. The numerical exploration of the solutions suggests that the system contains invariant surfaces. In particular, it can be proved that the surfaces $z = K x^{-r_3/r_1}$ are invariant in Ω^+. These surfaces are filled with periodic orbits enclosing the ray of quilibrium points P_λ (see [3]).

In order to determine these periodic orbits, we perturb the system around the equilibrium point $(r_2/a_{21} + \lambda a_{23}/a_{21}, r_1/a_{12}, \lambda)$,

$$\begin{aligned} x(t) &= \frac{r_2}{a_{21}} + \frac{a_{23}}{a_{21}} \lambda + \epsilon X(t)\,, \\ y(t) &= \frac{r_1}{a_{12}} + \epsilon Y(t)\,, \\ z(t) &= \lambda + \epsilon Z(t)\,, \end{aligned}$$

to obtain the perturbed system

$$\begin{aligned} \dot{X} &= -\frac{a_{12}}{a_{21}} (r_2 + a_{23}\lambda) Y - \epsilon\, a_{12} X Y\,, \\ \dot{Y} &= r_1 \frac{a_{21}}{a_{12}} X - r_1 \frac{a_{23}}{a_{12}} Z + \epsilon \left(a_{21} X Y - a_{23} Y Z \right)\,, \\ \dot{Z} &= a_{32} \lambda Z + \epsilon\, a_{32} Y Z\,. \end{aligned}$$

The introduction of stretched time variable $T = \omega t$ gives

$$\begin{aligned} \omega x' &= -\frac{a_{12}}{a_{21}}\left(r_2 + a_{23}\lambda\right) y - \epsilon\, a_{12} x y\,, \\ \omega y' &= r_1 \frac{a_{21}}{a_{12}} x - r_1 \frac{a_{23}}{a_{12}} z + \epsilon\left(a_{21} x y - a_{23} y z\right), \\ \omega z' &= a_{32}\lambda y + \epsilon\, a_{32} y z\,, \end{aligned} \tag{12}$$

after recalling $X = x$, $Y = y$ and $Z = z$ for the sake of simplicity. In order to apply the Poincaré–Lindstedt technique as described in section 2, we expand the solution of the system and ω as expressed in equations (3) and (4),

$$\begin{aligned} x(T) &= x_0(T) + \epsilon x_1(T) + \epsilon^2 x_2(T) + \cdots\,, \\ y(T) &= y_0(T) + \epsilon y_1(T) + \epsilon^2 y_2(T) + \cdots\,, \\ z(T) &= z_0(T) + \epsilon z_1(T) + \epsilon^2 z_2(T) + \cdots\,, \end{aligned}$$

and

$$\omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots\,.$$

By substituting these expasions into equation (12), and collecting terms in equal powers of ϵ, we get an equation for each order of the approximation. The order zero is given by

$$\begin{aligned} \omega_0 x_0' &= -\frac{a_{12}}{a_{21}}\left(r_2 + a_{23}\lambda\right) y_0\,, \\ \omega_0 y_0' &= r_1 \frac{a_{21}}{a_{12}} x_0 - r_1 \frac{a_{23}}{a_{12}} z_0\,, \\ \omega_0 z_0' &= a_{32}\lambda y_0\,. \end{aligned} \tag{13}$$

The solution to (13) is

$$\begin{aligned} x_0(T) &= \frac{a_{23}}{a_{21}} A + \frac{(r_2 + a_{23}\lambda) a_{12}}{a_{21}\omega_0} B\cos T - \frac{(r_2 + a_{23}\lambda) a_{12}}{a_{21}\omega_0} C\sin T\,, \\ y_0(T) &= C\cos T + B\sin T\,, \\ z_0(T) &= A - \frac{a_{32}\lambda}{\omega_0} B\cos T + \frac{a_{32}\lambda}{\omega_0} C\sin T\,, \end{aligned} \tag{14}$$

where A, B and C depend on the initial conditions of the solution. The derivatives of x_0, y_0 and z_0 are

$$\begin{aligned} x_0'(T) &= -\frac{(r_2 + a_{23}\lambda) a_{12}}{a_{21}\omega_0} C\cos T - \frac{(r_2 + a_{23}\lambda) a_{12}}{a_{21}\omega_0} B\sin T\,, \\ y_0'(T) &= B\cos T - C\sin T\,, \\ z_0'(T) &= \frac{a_{32}\lambda}{\omega_0} C\cos T + \frac{a_{32}\lambda}{\omega_0} B\sin T\,. \end{aligned} \tag{15}$$

The first order system is given by

$$\begin{aligned} \omega_0 x_1' + \frac{a_{12}}{a_{21}}\left(r_2 + a_{23}\lambda\right) y_1 &= -a_{12} x_0 y_0 - \omega_1 x_0'\,, \\ \omega_0 y_1' - r_1 \frac{a_{21}}{a_{12}} x_1 + r_1 \frac{a_{23}}{a_{12}} z_1 &= a_{21} x_0 y_0 - a_{23} y_0 z_0 - \omega_1 y_0'\,, \\ \omega_0 z_1' - a_{32}\lambda y_1 &= a_{32} y_0 z_0 - \omega_1 z_0'\,. \end{aligned}$$

The substitution of equations (14) and (15) in the first order system yields

$$
\begin{aligned}
\omega_0 x_1' + \frac{a_{12}}{a_{21}}\left(r_2 + a_{23}\lambda\right) y_1 &= \frac{a_{12}}{a_{21}\omega_0} C\left(\omega_1(r_2 + a_{23}\lambda) - \omega_0 a_{23} A\right)\cos T + \\
&+ \frac{a_{12}}{a_{21}\omega_0} B\left(\omega_1(r_2 + a_{23}\lambda) - \omega_0 a_{23} A\right)\sin T - \\
&- \frac{a_{12}^2 BC}{a_{21}\omega_0}(r_2 + a_{23}\lambda)\cos 2T + \frac{a_{12}^2}{2a_{21}\omega_0}\left((C^2 - B^2)(r_2 + a_{23}\lambda)\right)\sin 2T\,, \\
\omega_0 y_1' - \frac{a_{21} r_1}{a_{12}} x_1 + \frac{a_{23} r_1}{a_{12}} z_1 &= -\omega_1 B\cos T + \omega_1 C \sin T + \frac{BC}{\omega_0}\left(a_{12}(r_2 + a_{23}\lambda) + a_{23}a_{32}\lambda\right)\cos 2T + \\
&+ \frac{B^2 - C^2}{2\omega_0}\left(a_{12} r_2 + (a_{12} + a_{32}) a_{23}\lambda\right)\sin 2T\,, \\
\omega_0 z_1' - a_{32}\lambda y_1 &= \frac{a_{32}}{\omega_0} C(A\omega_0 - \omega_1\lambda)\cos T + \frac{a_{32}}{\omega_0} B(A\omega_0 - \omega_1\lambda)\sin T - \\
&- \frac{a_{32}^2\lambda}{\omega_0} BC\cos 2T + \frac{a_{32}^2\lambda}{2\omega_0}(C^2 - B^2)\sin 2T\,.
\end{aligned} \tag{16}
$$

The application of proposition 1 to the system (16) yields

$$
\omega_1 = -\frac{r_1 a_{23} A\left(a_{12} - a_{32}\right)\omega_0}{a_{12}(\omega_0^2 + r_1 r_2) + \lambda r_1 a_{23}(a_{12} + a_{32})}\,.
$$

From the analysis of this example, it follows that the application of the Poincaré–Lindstedt method involves working with expressions and procedures that should be automated to avoid mistakes in the algebraic manipulation of those developments.

4. Symbolic Algorithm for the Poincaré–Lindstedt Method

In this section, we present a general algorithm to compute periodic solutions through the application of the Poincaré–Lindstedt method to the system given in equation (1). Here, $\mathbb{P}$ refers to the set of Poisson series of the type defined in reference [10].

1. Define $X(\rho_1, \rho_2, \rho_3, q) \in \mathbb{P}$ for each $\rho_1, \rho_2, \rho_3, q \in \mathbb{N}$ such that $0 \leq \rho_1, \rho_2, \rho_3 \leq M$ and $0 \leq q \leq Q$, Q being the order of the expansion. Here,
$$X(\rho_1, \rho_2, \rho_3, q) = (x^{\rho_1} y^{\rho_2} z^{\rho_3})_q\,.$$
2. Define $DX(q), DY(q), DZ(q) \in \mathbb{P}$ for each $q \in \mathbb{N}$ such that $0 \leq q \leq Q$, Q being the order of the expansion. Here,
$$DX(q) = \frac{d}{dT} x_q\,, \quad DY(q) = \frac{d}{dT} y_q\,, \quad DZ(q) = \frac{d}{dT} z_q\,.$$
3. Define the array $W(q) \in \mathbb{P}$, for each $0 \leq q \leq Q$, to represent the coefficient ω_q.
4. The functions $f_1(x,y,z)$, $f_2(x,y,z)$ and $f_3(x,y,z)$ are represented by the following $(1+M)\times(1+M)\times(1+M)$ real hypermatrices,
$$F_1(\nu_1,\nu_2,\nu_3) = \left(\ f_{1,\nu_1\nu_2\nu_3}\ \right), \quad F_2(\nu_1,\nu_2,\nu_3) = \left(\ f_{2,\nu_1\nu_2\nu_3}\ \right), \quad F_3(\nu_1,\nu_2,\nu_3) = \left(\ f_{3,\nu_1\nu_2\nu_3}\ \right).$$
If $f_1(x,y,z)$, $f_2(x,y,z)$ and $f_3(x,y,z)$ are given by equation (2), then $f_{1,\nu_1,\nu_2,\nu_3} = 0$, $f_{2,\nu_1,\nu_2,\nu_3} = 0$ and $f_{3,\nu_1,\nu_2,\nu_3} = 0$, if $\nu_1 + \nu_2 + \nu_3 > M$. We will refer to the element f_{1,ν_1,ν_2,ν_3}, f_{2,ν_1,ν_2,ν_3} and f_{3,ν_1,ν_2,ν_3} as $F_1(\nu_1,\nu_2,\nu_3)$, $F_2(\nu_1,\nu_2,\nu_3)$ and $F_3(\nu_1,\nu_2,\nu_3)$ respectively.

4.1. 0–th Order Solution

1. Set $X(0,0,0,0) = 1$.
2. Set $W(0) = \sqrt{\alpha_{12}\alpha_{21} + \alpha_{23}\alpha_{32}}$. This means that $\omega_0 = \sqrt{\alpha_{12}\alpha_{21} + \alpha_{23}\alpha_{32}}$.

3. Compute $X(1,0,0,0)$, $X(0,1,0,0)$ and $X(0,0,1,0)$ as the solution to the unperturbed system (6). Notice that $X(1,0,0,0)$, $X(0,1,0,0)$ and $X(0,0,1,0)$ are modified Poisson series containing parameters with undetermined value corresponding to the integration constants.
4. Compute

$$DX(0) = \frac{d}{dT}X(1,0,0,0)\,, \quad DY(0) = \frac{d}{dT}X(0,1,0,0)\,, \quad DZ(0) = \frac{d}{dT}X(0,0,1,0)\,.$$

5. Calculate, for each ρ such that $2 \leq \rho \leq M$,

$$\begin{aligned} X(\rho,0,0,0) &= X(1,0,0,0)X(\rho-1,0,0,0)\,, \\ X(0,\rho,0,0) &= X(0,1,0,0)X(0,\rho-1,0,0)\,, \\ X(0,0,\rho,0) &= X(0,0,1,0)X(0,0,\rho-1,0)\,, \end{aligned}$$

that is, x_0^ρ, y_0^ρ and z_0^ρ.
6. Compute, for each ρ_1, ρ_2, ρ_3 such that $1 \leq \rho_1, \rho_2, \rho_3 \leq M$, the modified Poisson series $(x^{\rho_1}y^{\rho_2}z^{\rho_3})_0$ through

$$X(\rho_1,\rho_2,\rho_3,0) = X(\rho_1,0,0,0)X(0,\rho_2,0,0)X(0,0,\rho_3,0)\,.$$

4.2. 1–th Order Solution

1. Compute the first term of the right–hand side of equation (7),

$$\begin{aligned} U_{1,1} &= \sum_{0\leq q\leq M}\sum_{0\leq \nu_1\leq q}\sum_{0\leq \nu_2\leq \nu_1} f_{1,\nu_2,q-\nu_1,\nu_1-\nu_2} x_0^{\nu_2} y_0^{q-\nu_1} z_0^{\nu_1-\nu_2}\,, \\ U_{2,1} &= \sum_{0\leq q\leq M}\sum_{0\leq \nu_1\leq q}\sum_{0\leq \nu_2\leq \nu_1} f_{2,\nu_2,q-\nu_1,\nu_1-\nu_2} x_0^{\nu_2} y_0^{q-\nu_1} z_0^{\nu_1-\nu_2}\,, \\ U_{3,1} &= \sum_{0\leq q\leq M}\sum_{0\leq \nu_1\leq q}\sum_{0\leq \nu_2\leq \nu_1} f_{3,\nu_2,q-\nu_1,\nu_1-\nu_2} x_0^{\nu_2} y_0^{q-\nu_1} z_0^{\nu_1-\nu_2}\,. \end{aligned}$$

These series are calculated as

$$\begin{aligned} U_{1,1} &= \sum_{0\leq q\leq M}\sum_{0\leq \nu_1\leq q}\sum_{0\leq \nu_2\leq \nu_1} F_1(\nu_2, q-\nu_1, \nu_1-\nu_2)X(\nu_2, q-\nu_1, \nu_1-\nu_2, 0)\,, \\ U_{2,1} &= \sum_{0\leq q\leq M}\sum_{0\leq \nu_1\leq q}\sum_{0\leq \nu_2\leq \nu_1} F_2(\nu_2, q-\nu_1, \nu_1-\nu_2)X(\nu_2, q-\nu_1, \nu_1-\nu_2, 0)\,, \\ U_{3,1} &= \sum_{0\leq q\leq M}\sum_{0\leq \nu_1\leq q}\sum_{0\leq \nu_2\leq \nu_1} F_3(\nu_2, q-\nu_1, \nu_1-\nu_2)X(\nu_2, q-\nu_1, \nu_1-\nu_2, 0)\,. \end{aligned}$$

2. Calculate the rest of the right–hand side of (7),

$$R_1 = -\omega_1 x_0'\,, \quad S_1 = -\omega_1 y_0'\,, \quad T_1 = -\omega_1 z_0'\,,$$

through

$$R_1 = -W(1)DX(0)\,, \quad S_1 = -W(1)DY(0)\,, \quad T_1 = -W(1)DZ(0)\,,$$

The value of ω_1 is fitted to assure that no secular terms are included in the solution. To do that, we apply the proposition 1.
3. Once $W(1)$ has been determined, we substitute its value in equation (7). This is equivalent to eliminating resonant terms from the right–hand side of this equation. Now, we calculate $X(1,0,0,1)$, $X(0,1,0,1)$ and $X(0,0,1,1)$ as the solution to system (7) without resonant terms.

4. Compute
$$DX(1) = \frac{d}{dT}X(1,0,0,1)\,, \quad DY(1) = \frac{d}{dT}X(0,1,0,1)\,, \quad DZ(1) = \frac{d}{dT}X(0,0,1,1)\,,$$
that is, x_1', y_1' and z_1'.
5. Calculate, for each ρ such that $2 \le \rho \le M$,
$$\begin{aligned} X(\rho,0,0,1) &= \sum_{0\le\nu\le1} X(\rho-1,0,0,\nu)X(1,0,0,1-\nu)\,, \\ X(0,\rho,0,1) &= \sum_{0\le\nu\le1} X(0,\rho-1,0,\nu)X(0,1,0,1-\nu)\,, \\ X(0,0,\rho,1) &= \sum_{0\le\nu\le1} X(0,0,\rho-1,\nu)X(0,0,1,1-\nu)\,, \end{aligned}$$
that is, $(x^\rho)_1$, $(y^\rho)_1$ and $(z^\rho)_1$.
6. Compute, for each ρ_1,ρ_2,ρ_3 such that $1 \le \rho_1,\rho_2,\rho_3 \le M$, the modified Poisson series $(x^{\rho_1}y^{\rho_2}z^{\rho_3})_1$ through
$$X(\rho_1,\rho_2,\rho_3,1) = \sum_{0\le\nu_1\le1}\sum_{0\le\nu_2\le\nu_1} X(\rho_1,0,0,\nu_1-\nu_2)X(0,\rho_2,0,\nu_2)X(0,0,\rho_3,1-\nu_1)\,.$$

4.3. p–th Order Solution, for $p > 1$

1. Compute the following part of the right–hand side of equation (8),
$$\begin{aligned} U_{1,p} &= \sum_{0\le q\le M}\sum_{0\le\nu_1\le q}\sum_{0\le\nu_2\le\nu_1} f_{1,\nu_2,q-\nu_1,\nu_1-\nu_2}\left[x^{\nu_2}y^{q-\nu_1}z^{\nu_1-\nu_2}\right]_{p-1} - \sum_{1\le\nu\le p-1} x_\nu'\omega_{p-\nu}\,, \\ U_{2,p} &= \sum_{0\le q\le M}\sum_{0\le\nu_1\le q}\sum_{0\le\nu_2\le\nu_1} f_{2,\nu_2,q-\nu_1,\nu_1-\nu_2}\left[x^{\nu_2}y^{q-\nu_1}z^{\nu_1-\nu_2}\right]_{p-1} - \sum_{1\le\nu\le p-1} y_\nu'\omega_{p-\nu}\,, \\ U_{3,p} &= \sum_{0\le q\le M}\sum_{0\le\nu_1\le q}\sum_{0\le\nu_2\le\nu_1} f_{3,\nu_2,q-\nu_1,\nu_1-\nu_2}\left[x^{\nu_2}y^{q-\nu_1}z^{\nu_1-\nu_2}\right]_{p-1} - \sum_{1\le\nu\le p-1} z_\nu'\omega_{p-\nu}\,, \end{aligned}$$
which corresponds to
$$\begin{aligned} U_{1,p} &= \sum_{0\le q\le M}\sum_{0\le\nu_1\le q}\sum_{0\le\nu_2\le\nu_1} F_1(\nu_2,q-\nu_1,\nu_1-\nu_2)X(\nu_2,q-\nu_1,\nu_1-\nu_2,p-1) - \sum_{1\le\nu\le p-1} DX(\nu)W(p-\nu)\,, \\ U_{2,p} &= \sum_{0\le q\le M}\sum_{0\le\nu_1\le q}\sum_{0\le\nu_2\le\nu_1} F_2(\nu_2,q-\nu_1,\nu_1-\nu_2)X(\nu_2,q-\nu_1,\nu_1-\nu_2,p-1) - \sum_{1\le\nu\le p-1} DY(\nu)W(p-\nu)\,, \\ U_{3,p} &= \sum_{0\le q\le M}\sum_{0\le\nu_1\le q}\sum_{0\le\nu_2\le\nu_1} F_3(\nu_2,q-\nu_1,\nu_1-\nu_2)X(\nu_2,q-\nu_1,\nu_1-\nu_2,p-1) - \sum_{1\le\nu\le p-1} DZ(\nu)W(p-\nu)\,. \end{aligned}$$
2. Calculate the rest of the right–hand side of (7),
$$R_p = -\omega_p x_0'\,, \quad S_p = -\omega_p y_0'\,, \quad T_p = -\omega_p z_0'\,.$$
The value of ω_p is fitted to assure that resonance disappears. As before, we apply the proposition 1. Then ω_p must be fitted to make resonance disappear, and we can compute $W(p)$ as a modified Poisson series depending on the integration constants of the problem.
3. Once $W(p)$ has been determined, we substitute it in equation (8). This corresponds to eliminating resonant terms from the right–hand side of this equation. Now, we calculate $X(1,0,0,p)$, $X(0,1,0,p)$ and $X(0,0,1,p)$ as the solution to system (8) without resonant terms. For that purpose we use the undetermined coefficients method.
4. Compute
$$DX(p) = \frac{d}{dT}X(1,0,0,p)\,, \quad DY(p) = \frac{d}{dT}X(0,1,0,p)\,, \quad DZ(p) = \frac{d}{dT}X(0,0,1,p)\,.$$

5. Determine, for each ρ such that $2 \leq \rho \leq M$,

$$X(\rho,0,0,p) = \sum_{0\leq\nu\leq p} X(\rho-1,0,0,\nu)X(1,0,0,p-\nu)\,,$$
$$X(0,\rho,0,p) = \sum_{0\leq\nu\leq p} X(0,\rho-1,0,\nu)X(0,1,0,p-\nu)\,,$$
$$X(0,0,\rho,p) = \sum_{0\leq\nu\leq p} X(0,0,\rho-1,\nu)X(0,0,1,p-\nu)\,,$$

that is $(x^\rho)_p$, $(y^\rho)_p$ and $(z^\rho)_p$.

6. Compute, for each ρ_1, ρ_2, ρ_3 such that $1 \leq \rho_1, \rho_2, \rho_3 \leq M$, the modified Poisson series $(x^{\rho_1} y^{\rho_2} z^{\rho_3})_p$ through

$$X(\rho_1,\rho_2,\rho_3,p) = \sum_{0\leq\nu_1\leq p}\sum_{0\leq\nu_2\leq\nu_1} X(\rho_1,0,0,\nu_1-\nu_2)X(0,\rho_2,0,\nu_2)X(0,0,\rho_3,p-\nu_1)\,.$$

The approximation to the periodic solution to (1) is given by

$$\begin{aligned}
x(t) &= X(1,0,0,0)(T) + \epsilon X(1,0,0,1)(T) + \epsilon^2 X(1,0,0,2)(T) + \cdots + \epsilon^Q X(1,0,0,Q)\,,\\
y(t) &= X(0,1,0,0)(T) + \epsilon X(0,1,0,1)(T) + \epsilon^2 X(0,1,0,2)(T) + \cdots + \epsilon^Q X(0,1,0,Q)\,,\\
z(t) &= X(0,0,1,0)(T) + \epsilon X(0,0,1,1)(T) + \epsilon^2 X(0,0,1,2)(T) + \cdots + \epsilon^Q X(0,0,1,Q)\,,
\end{aligned}$$

where $X(1,0,0,q)(T)$, $X(0,1,0,q)(T)$ and $X(0,0,1,q)(T)$ $(1 \leq q \leq Q)$ are 2π–periodic in $T = \omega t$, and

$$\omega = W(0) + \epsilon W(1) + \epsilon^2 W(2) + \cdots + \epsilon^Q W(Q)\,.$$

In the following section, we apply this algorithm to a particular case of the Lotka–Volterra model described in section 3.

5. Computation of a periodic orbit in a three–dimensional Lotka–Volterra model

Let us consider the system

$$\begin{aligned}
\dot{x} &= x(1-y)\,,\\
\dot{y} &= y(-2+x-z)\,,\\
\dot{z} &= z(-1+y)\,.
\end{aligned} \tag{17}$$

The equilibrium points of the system are $(2+\lambda, 1, \lambda)$, $\lambda \in \mathbb{R}^+$. Let us take, for instance, $\lambda = 1/2$, and the equilibrium point $(5/2, 1, 1/2)$. Following the procedure employed in section 3, equation (17) is transformed into

$$\begin{aligned}
\omega x' &= -\frac{5}{2}y - \epsilon yz\,,\\
\omega y' &= x - z + \epsilon(xy - yz)\,,\\
\omega z' &= \frac{1}{2}y + \epsilon yz\,.
\end{aligned} \tag{18}$$

We compute ω_0 from the unperturbed system $(\epsilon = 0)$,

$$\omega_0 = \sqrt{3}\,.$$

By substituting (3) and (4) into equation (18), and collecting terms in equal powers of ϵ, we get an equation for each order of the series solution. The unperturbed system is given by

$$\begin{aligned}
\omega_0 x_0' + \frac{5}{2}y_0 &= 0\,,\\
\omega_0 y_0' - x_0 + z_0 &= 0\,,\\
\omega_0 z_0' - \frac{1}{2}y_0 &= 0\,.
\end{aligned}$$

Table 1. Zero order approximation of the solution to (18).

nT	$x_0(\cos)$	$x_0(\sin)$	$y_0(\cos)$	$y_0(\sin)$	$z_0(\cos)$	$z_0(\sin)$
0	$\frac{7}{6}$	0	0	0	$\frac{7}{6}$	0
T	$\frac{5}{6}$	$-\frac{5\sqrt{3}}{6}$	1	$\frac{\sqrt{3}}{3}$	$-\frac{1}{6}$	$\frac{\sqrt{3}}{6}$

The solution to this problem, with initial conditions $x_0(0) = 2$, $y_0(0) = 1$, $z_0(0) = 1$, is

$$\begin{aligned} x_0(T) &= \frac{7}{6} + \frac{5}{6}\cos T - \frac{5\sqrt{3}}{6}\sin T\,, \\ y_0(T) &= \cos T + \frac{\sqrt{3}}{3}\sin T\,, \\ z_0(T) &= \frac{7}{6} - \frac{1}{6}\cos T + \frac{\sqrt{3}}{6}\sin T\,. \end{aligned} \tag{19}$$

In table 1, we give the solution expressed in a matrix. This notation will be very useful to express the approximation to the solution for orders larger than two.

The first order system is

$$\begin{aligned} \omega_0 x_1' + \frac{5}{2}y_1 &= -x_0 y_0 - \omega_1 x_0'\,, \\ \omega_0 y_1' - x_1 + z_1 &= x_0 y_0 - y_0 z_0 - \omega_1 y_0'\,, \\ \omega_0 z_1' - \frac{1}{2}y_1 &= y_0 z_0 - \omega_1 z_0'\,. \end{aligned} \tag{20}$$

The substitution of x_0, y_0 and z_0, given in equation (19), into equation (20) produces the system

$$\begin{aligned} \omega_0 x_1' + \frac{5}{2}y_1 &= \left(\frac{5\sqrt{3}}{6}\omega_1 - \frac{7}{6}\right)\cos T + \left(\frac{5}{6}\omega_1 - \frac{7\sqrt{3}}{18}\right)\sin T - \frac{5}{6}\cos 2T + \frac{5\sqrt{3}}{18}\sin 2T\,, \\ \omega_0 y_1' - x_1 + z_1 &= -\frac{\sqrt{3}}{3}\omega_1\cos T + \omega_1\sin T + \cos 2T - \frac{\sqrt{3}}{3}\sin 2T\,, \\ \omega_0 z_1' - \frac{1}{2}y_1 &= \left(\frac{7}{6} - \frac{\sqrt{3}}{6}\omega_1\right)\cos T + \left(\frac{7\sqrt{3}}{18} - \frac{1}{6}\omega_1\right)\sin T - \frac{1}{6}\cos 2T + \frac{\sqrt{3}}{18}\sin 2T\,. \end{aligned} \tag{21}$$

In order to eliminate secular terms in equation (21), the following system of equations must be satisfied (proposition 1):

$$\begin{aligned} \sqrt{3}\left(-\frac{\sqrt{3}}{3}\omega_1\right) + \left(\frac{7\sqrt{3}}{18} - \frac{1}{6}\omega_1\right) - \left(\frac{5}{6}\omega_1 - \frac{7\sqrt{3}}{18}\right) &= 0\,, \\ \sqrt{3}\omega_1 - \left(\frac{7}{6} - \frac{\sqrt{3}}{6}\omega_1\right) + \left(\frac{5\sqrt{3}}{6}\omega_1 - \frac{7}{6}\right) &= 0\,. \end{aligned} \tag{22}$$

The solution to the system (22) is

$$\omega_1 = \frac{7\sqrt{3}}{18}\,.$$

The substitution of ω_1 into equation (21) yields

$$\begin{aligned} \omega_0 x_1' + \frac{5}{2}y_1 &= -\frac{7}{36}\cos T - \frac{7\sqrt{3}}{108}\sin T - \frac{5}{6}\cos 2T + \frac{5\sqrt{3}}{18}\sin 2T\,, \\ \omega_0 y_1' - x_1 + z_1 &= -\frac{7}{18}\cos T + \frac{7\sqrt{3}}{18}\sin T + \cos 2T - \frac{\sqrt{3}}{3}\sin 2T\,, \\ \omega_0 z_1' - \frac{1}{2}y_1 &= \frac{35}{36}\cos T + \frac{35\sqrt{3}}{108}\sin T - \frac{1}{6}\cos 2T + \frac{\sqrt{3}}{18}\sin 2T\,. \end{aligned} \tag{23}$$

The solution to (23) can be written as

$$\begin{aligned} x_1(T) &= \frac{7}{108}\cos T - \frac{7\sqrt{3}}{108}\sin T + \frac{35}{324}\cos 2T - \frac{858\sqrt{3}}{324}\sin 2T\,, \\ y_1(T) &= \frac{8}{27}\cos 2T + \frac{16}{81}\sqrt{3}\sin 2T\,, \\ z_1(T) &= -\frac{35}{108}\cos T + \frac{35\sqrt{3}}{108}\sin T - \frac{25}{324}\cos 2T - \frac{\sqrt{3}}{324}\sin 2T\,. \end{aligned} \tag{24}$$

Table 2. First order approximation of the solution to (18).

nT	$x_1(\cos)$	$x_1(\sin)$	$y_1(\cos)$	$y_1(\sin)$	$z_1(\cos)$	$z_1(\sin)$
0	0	0	0	0	0	0
T	$\frac{7}{108}$	$-\frac{7\sqrt{3}}{108}$	0	0	$-\frac{35}{108}$	$\frac{35\sqrt{3}}{108}$
$2T$	$\frac{35}{324}$	$-\frac{858\sqrt{3}}{324}$	$\frac{8}{27}$	$\frac{16\sqrt{3}}{81}$	$-\frac{25}{324}$	$-\frac{\sqrt{3}}{324}$

In table 2, we show the first order approximation to the solution.

The system of second order is arranged as

$$\begin{aligned} \omega_0 x_2' + \frac{5}{2} y_2 &= -x_1 y_0 - x_0 y_1 - \omega_1 x_1' - \omega_2 x_0' \,, \\ \omega_0 y_2' - x_2 + z_2 &= x_1 y_0 + x_0 y_1 - z_1 y_0 - z_0 y_1 - \omega_1 y_1' - \omega_2 y_0' \,, \\ \omega_0 z_2' - \frac{1}{2} y_2 &= y_1 z_0 + y_0 z_1 - \omega_1 z_1' - \omega_2 z_0' \,. \end{aligned} \tag{25}$$

By substituting ω_1, x_0, y_0, z_0, x_1, y_1, z_1 and their derivatives into equation (25), we get

$$\begin{aligned} \omega_0 x_2' + \frac{5}{2} y_2 &= -\frac{11161}{648}\cos T + \frac{5}{6}\omega_2\sqrt{3}\cos T - \frac{11401}{1944}\sqrt{3}\sin T + \frac{5}{6}\omega_2 \sin T - \\ &- \frac{5977}{486}\cos 2T + \frac{5893}{1458}\sqrt{3}\sin 2T - \frac{5}{9}\cos 3T + \frac{25}{162}\sqrt{3}\sin 3T \,, \\ \omega_0 y_2' - x_2 + z_2 &= -\frac{5}{27}\cos T - \frac{1}{3}\omega_2\sqrt{3}\cos T + \frac{7}{81}\sqrt{3}\sin T + \omega_2 \sin T + \\ &+ \frac{7255}{486}\cos 2T - \frac{2381}{486}\sqrt{3}\sin 2T + \frac{2}{3}\cos 3T - \frac{4}{27}\sqrt{3}\sin 3T \,, \\ \omega_0 z_2' - \frac{1}{2} y_2 &= \frac{3695}{216}\cos T - \frac{1}{6}\omega_2\sqrt{3}\cos T + \frac{3679}{648}\sqrt{3}\sin T - \frac{1}{6}\omega_2 \sin T - \\ &- \frac{1201}{486}\cos 2T + \frac{1621}{1458}\sqrt{3}\sin 2T - \frac{1}{9}\cos 3T - \frac{1}{162}\sqrt{3}\sin 3T \,. \end{aligned} \tag{26}$$

Now, if we apply the proposition 1 to equation (26) to eliminate secular terms, we get the system

$$\begin{aligned} \sqrt{3}\left(-\frac{5}{27} - \frac{\sqrt{3}\omega_2}{3}\right) + \left(\frac{3679\sqrt{3}}{648} - \frac{\omega_2}{6}\right) - \left(\frac{5\omega_2}{6} - \frac{11401\sqrt{3}}{1944}\right) &= 0 \,, \\ \sqrt{3}\left(\frac{7\sqrt{3}}{81} + \omega_2\right) - \left(\frac{3695}{216} - \frac{\sqrt{3}\omega_2}{6}\right) + \left(\frac{5\sqrt{3}\omega_2}{6} - \frac{11161}{648}\right) &= 0 \,, \end{aligned} \tag{27}$$

with solution

$$\omega_2 = \frac{11039\sqrt{3}}{1944} \,.$$

The solution to the second order system obtained after the substitution of ω_2 into equation (26) yields

$$\begin{aligned} x_2(T) &= \frac{3635}{972} + \frac{13211}{11664}\cos T - \frac{11771}{11664}\sqrt{3}\sin T + \frac{1253}{729}\cos 2T - \frac{2818}{729}\sqrt{3}\sin 2T + \frac{145}{11664}\cos 3T - \frac{20}{243}\sqrt{3}\sin 3T \,, \\ y_2(T) &= \frac{353}{81}\cos 2T + \frac{727}{243}\sqrt{3}\sin 2T + \frac{2}{27}\cos 3T + \frac{149}{1944}\sqrt{3}\sin 3T \,, \\ z_2(T) &= \frac{3635}{972} - \frac{55183}{11664}\cos T + \frac{55471}{11664}\sqrt{3}\sin T - \frac{1901}{1458}\cos 2T - \frac{71}{1458}\sqrt{3}\sin 2T - \frac{125}{11664}\cos 3T - \frac{2}{243}\sqrt{3}\sin 3T \,. \end{aligned}$$

The second order approximation to the solution is given in table 3.

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Table 3. Second order approximation of the solution to (18).

nT	$x_2(\cos)$	$x_2(\sin)$	$y_2(\cos)$	$y_2(\sin)$	$z_2(\cos)$	$z_2(\sin)$
0	$\frac{3635}{972}$	0	0	0	$\frac{3635}{972}$	0
T	$\frac{13211}{11664}$	$-\frac{11771\sqrt{3}}{11664}$	0	0	$-\frac{55183}{11664}$	$\frac{55471\sqrt{3}}{11664}$
$2T$	$\frac{1253}{729}$	$-\frac{2818\sqrt{3}}{729}$	$\frac{353}{81}$	$\frac{727\sqrt{3}}{243}$	$-\frac{1901}{1458}$	$-\frac{71\sqrt{3}}{1458}$
$3T$	$\frac{145}{11664}$	$-\frac{20\sqrt{3}}{243}$	$\frac{2}{27}$	$\frac{149\sqrt{3}}{1944}$	$-\frac{125}{11664}$	$-\frac{2\sqrt{3}}{243}$

Table 4. Third order approximation of the solution to (18).

nT	$x_3(\cos)$	$x_3(\sin)$	$y_3(\cos)$	$y_3(\sin)$	$z_3(\cos)$	$z_3(\sin)$
0	0	0	0	0	0	0
T	$\frac{5288995}{209952}$	$-\frac{4519459\sqrt{3}}{209952}$	0	0	$-\frac{20637263}{209952}$	$\frac{20800847\sqrt{3}}{209952}$
$2T$	$\frac{69256801}{1889568}$	$-\frac{150887471\sqrt{3}}{1889568}$	$\frac{7081553}{78732}$	$\frac{14758267\sqrt{3}}{236196}$	$-\frac{53688791}{1889568}$	$-\frac{1933703\sqrt{3}}{1889568}$
$3T$	$\frac{91733}{209952}$	$-\frac{5309\sqrt{3}}{2187}$	$\frac{173}{81}$	$\frac{13459\sqrt{3}}{5832}$	$-\frac{71833}{209952}$	$-\frac{575\sqrt{3}}{2187}$
$4T$	$-\frac{1381}{1889568}$	$-\frac{49643\sqrt{3}}{1889568}$	$\frac{3739}{196830}$	$\frac{14947\sqrt{3}}{590490}$	$\frac{3631}{9447840}$	$-\frac{21727\sqrt{3}}{9447840}$

The complete solution to the second order of approximation to (18) is

$$
\begin{aligned}
x(t) &= \left(\frac{7}{6}+\frac{5}{6}\cos\omega t-\frac{5\sqrt{3}}{6}\sin\omega t\right)+\epsilon\left(\frac{7}{108}\cos\omega t-\frac{7\sqrt{3}}{108}\sin\omega t+\frac{35}{324}\cos 2\omega t-\frac{858\sqrt{3}}{324}\sin 2\omega t\right)+\\
&+ \epsilon^2\left(\frac{3635}{972}+\frac{13211}{11664}\cos\omega t-\frac{11771}{11664}\sqrt{3}\sin\omega t+\frac{1253}{729}\cos 2\omega t-\frac{2818}{729}\sqrt{3}\sin 2\omega t+\right.\\
&\qquad \left.+\frac{145}{11664}\cos 3\omega t-\frac{20}{243}\sqrt{3}\sin 3\omega t\right),
\end{aligned}
$$

$$
\begin{aligned}
y(t) &= \left(\cos\omega t+\frac{\sqrt{3}}{3}\sin\omega t\right)+\epsilon\left(\frac{8}{27}\cos 2\omega t+\frac{16}{81}\sqrt{3}\sin 2\omega t\right)+\\
&+ \epsilon^2\left(\frac{353}{81}\cos 2\omega t+\frac{727}{243}\sqrt{3}\sin 2\omega t+\frac{2}{27}\cos 3\omega t+\frac{149}{1944}\sqrt{3}\sin 3\omega t\right),
\end{aligned}
$$

$$
\begin{aligned}
z(t) &= \left(\frac{7}{6}-\frac{1}{6}\cos\omega t+\frac{\sqrt{3}}{6}\sin\omega t\right)+\epsilon\left(\frac{-35}{108}\cos\omega t+\frac{35\sqrt{3}}{108}\sin\omega t-\frac{25}{324}\cos 2\omega t-\frac{\sqrt{3}}{324}\sin 2\omega t\right)+\\
&+ \epsilon^2\left(\frac{3635}{972}-\frac{55183}{11664}\cos\omega t+\frac{55471}{11664}\sqrt{3}\sin\omega t-\frac{1901}{1458}\cos 2\omega t-\frac{71}{1458}\sqrt{3}\sin 2\omega t-\right.\\
&\qquad \left.-\frac{125}{11664}\cos 3\omega t-\frac{2}{243}\sqrt{3}\sin 3\omega t\right),
\end{aligned}
$$

where

$$
\omega=\sqrt{3}+\epsilon\frac{7\sqrt{3}}{18}+\epsilon^2\frac{11039\sqrt{3}}{1944}.
$$

In Figure 1, we represent the approximate solution to (18) up to second order. We have applied the algorithm up to the fifth order of the perturbation method. In tables 4, 5 and 6, we show the corresponding approximations to the solution.

In Figures 2, 3 and 4, we compare between the numerical solution to equation (17) with initial conditions

$$
x(0)=2.864670867\,,\quad y(0)=1.187254264\,,\quad z(0)=0.669094392\,,
$$

obtained by using a fourth order Runge–Kutta method, and a fifth order approximation to the solution computed through the Poincaré–Lindstedt method.

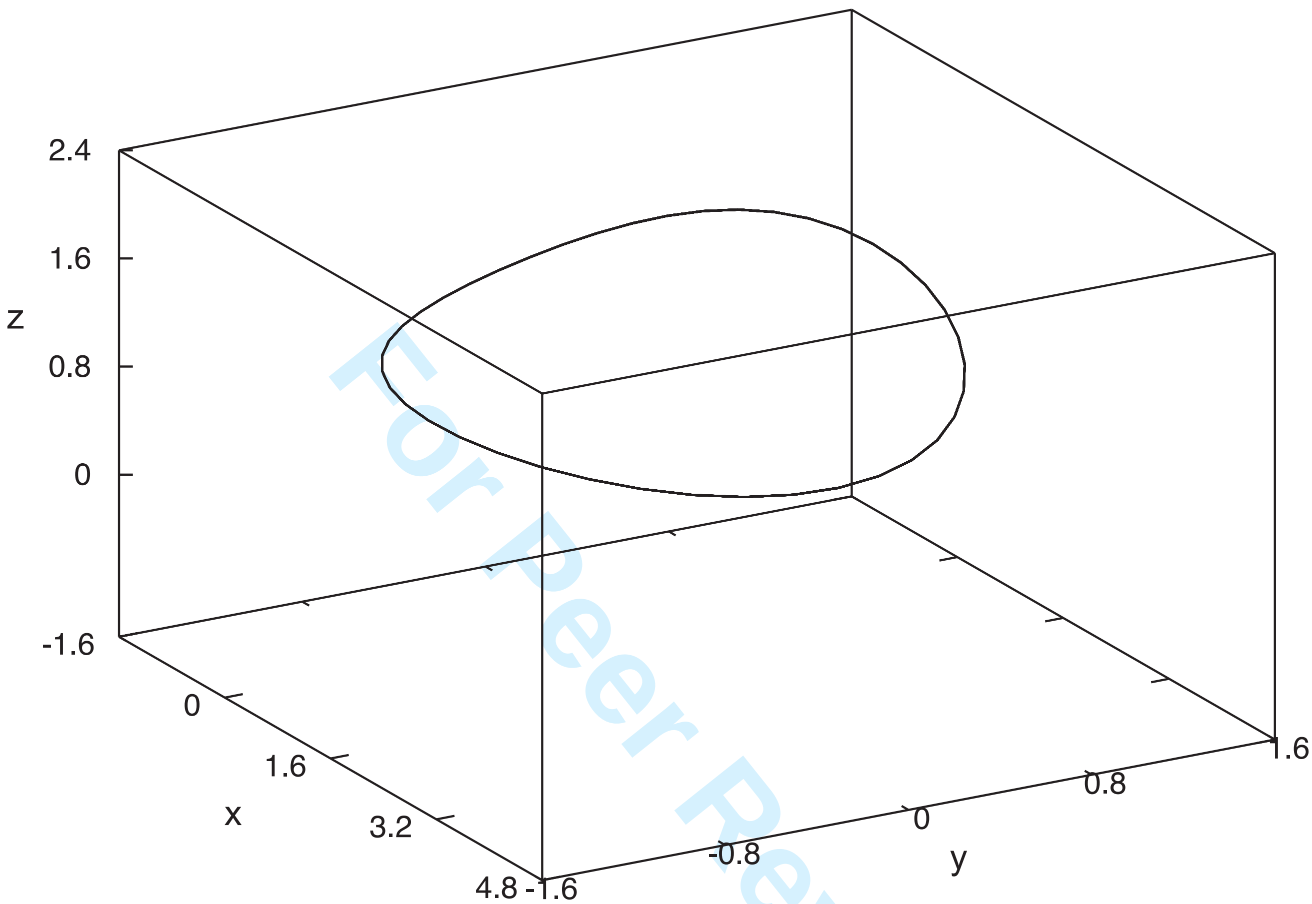

Figure 1. Second order of approximation to the periodic solution to (18) with initial conditions $x(0) = 2$, $y(0) = 1$ and $z(0) = 1$.

Table 5. Fourth order approximation of the solution to (18).

nT	$x_4(\cos)$	$x_4(\sin)$	$y_4(\cos)$	$y_4(\sin)$	$z_4(\cos)$	$z_4(\sin)$
0	0	0	0	0	0	0
T	$\frac{86193667007}{136048896}$	$-\frac{72161213471\sqrt{3}}{136048896}$	0	0	$-\frac{324836601907}{136048896}$	$\frac{327920266579\sqrt{3}}{136048896}$
$2T$	$\frac{7744178015}{8503056}$	$-\frac{16607791885\sqrt{3}}{8503056}$	$\frac{173125481}{78732}$	$\frac{362888423\sqrt{3}}{236196}$	$-\frac{6064029031}{8503056}$	$-\frac{213879211\sqrt{3}}{8503056}$
$3T$	$\frac{607384603}{45349632}$	$-\frac{64295279\sqrt{3}}{944784}$	$\frac{15506407}{262440}$	$\frac{2473590269\sqrt{3}}{37791360}$	$-\frac{2272853947}{226748160}$	$-\frac{7276265\sqrt{3}}{944784}$
$4T$	$-\frac{27673}{42515280}$	$-\frac{49468019\sqrt{3}}{42515280}$	$\frac{63515}{78732}$	$\frac{268507\sqrt{3}}{236196}$	$\frac{161017}{8503056}$	$-\frac{919021\sqrt{3}}{8503056}$
$5T$	$-\frac{1201055}{816293376}$	$-\frac{6904223\sqrt{3}}{816293376}$	$\frac{770183}{136048896}$	$\frac{3234361\sqrt{3}}{408146688}$	$\frac{489619}{816293376}$	$-\frac{293357\sqrt{3}}{816293376}$

Table 6. Fifth order approximation of the solution to (18).

nT	$x_5(\cos)$	$x_5(\sin)$	$y_5(\cos)$	$y_5(\sin)$	$z_5(\cos)$	$z_5(\sin)$
0	0	0	0	0	0	0
T	$\frac{42656273897017}{2448880128}$	$-\frac{35340801254905\sqrt{3}}{2448880128}$	0	0	$-\frac{157964924282093}{244888012}$	$\frac{159604905294125\sqrt{3}}{2448880128}$
$2T$	$\frac{60922435954207}{2448880128}$	$-\frac{129611403927857\sqrt{3}}{2448880128}$	$\frac{91175725952381}{1530550080}$	$\frac{191657222652271\sqrt{3}}{4591650240}$	$-\frac{26631698829493}{1360488960}$	$-\frac{556726364579\sqrt{3}}{816293376}$
$3T$	$\frac{1663756887337}{4081466880}$	$-\frac{20899847251\sqrt{3}}{10628820}$	$\frac{2666039413}{1574640}$	$\frac{107976263807\sqrt{3}}{56687040}$	$-\frac{1214302504493}{4081466880}$	$-\frac{2428905889\sqrt{3}}{10628820}$
$4T$	$\frac{3998460835}{5509980288}$	$-\frac{227140186651\sqrt{3}}{5509980288}$	$\frac{159359170223}{5739562800}$	$\frac{699889910399\sqrt{3}}{17218688400}$	$\frac{98775892471}{137749507200}$	$-\frac{548181165727\sqrt{3}}{137749507200}$
$5T$	$-\frac{1082591857}{14693280768}$	$-\frac{36726450101\sqrt{3}}{73466403840}$	$\frac{65050957}{204073344}$	$\frac{289260611\sqrt{3}}{612220032}$	$\frac{552444341}{14693280768}$	$-\frac{1626771191\sqrt{3}}{73466403840}$
$6T$	$-\frac{213946291}{257132413440}$	$-\frac{117096907\sqrt{3}}{42855402240}$	$\frac{11520667}{5952139200}$	$\frac{132844807\sqrt{3}}{53569252800}$	$\frac{225539761}{1285662067200}$	$-\frac{3970613\sqrt{3}}{214277011200}$

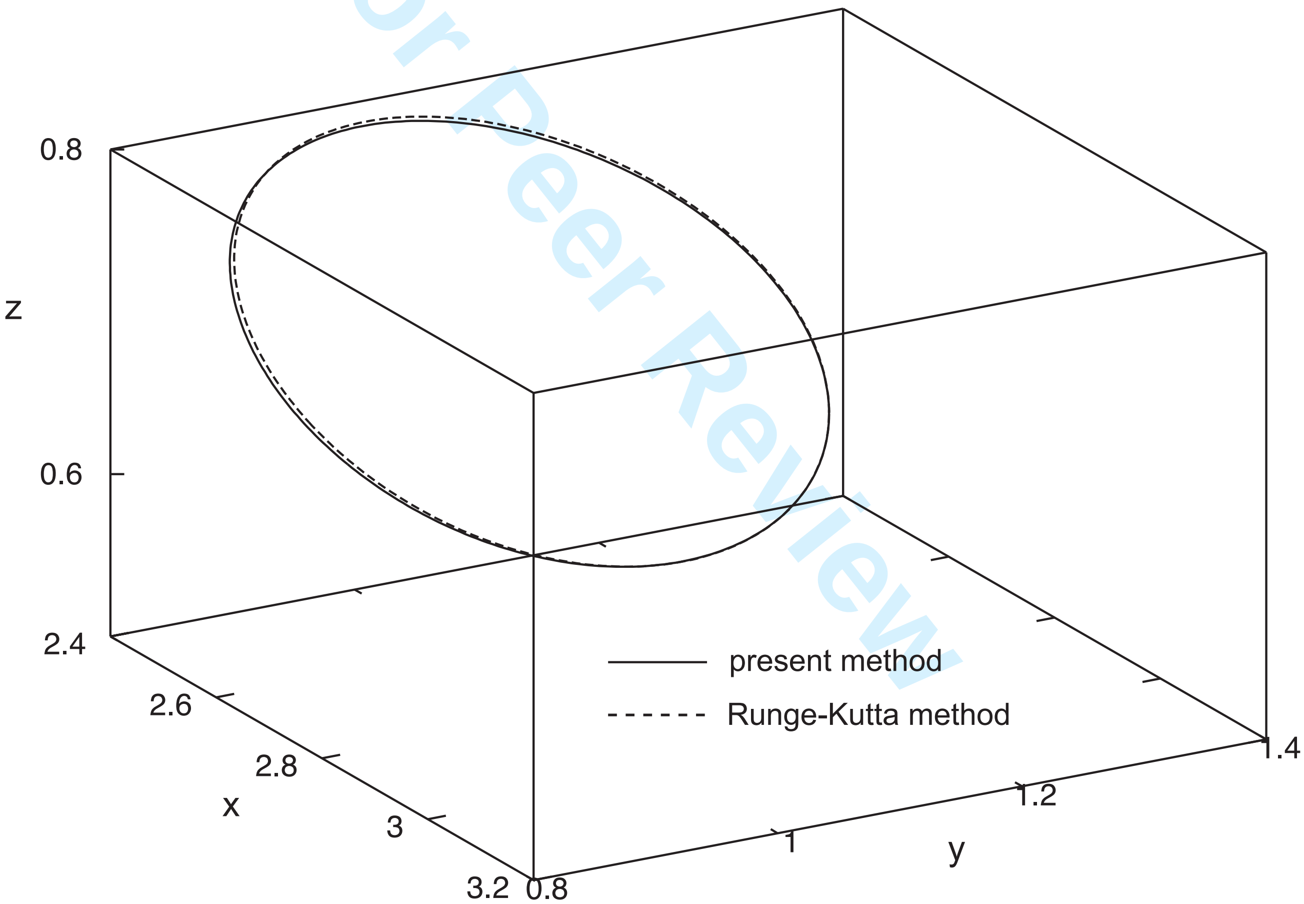

Figure 2. Comparison between the approximate solution to equation (17) with initial conditions $x(0) = 2.864670867$, $y(0) = 1.187254264$, $z(0) = 0.669094392$, obtained by using a fourth order Runge–Kutta method and a fifth order approximation to the solution computed through the Poincaré–Lindstedt method.

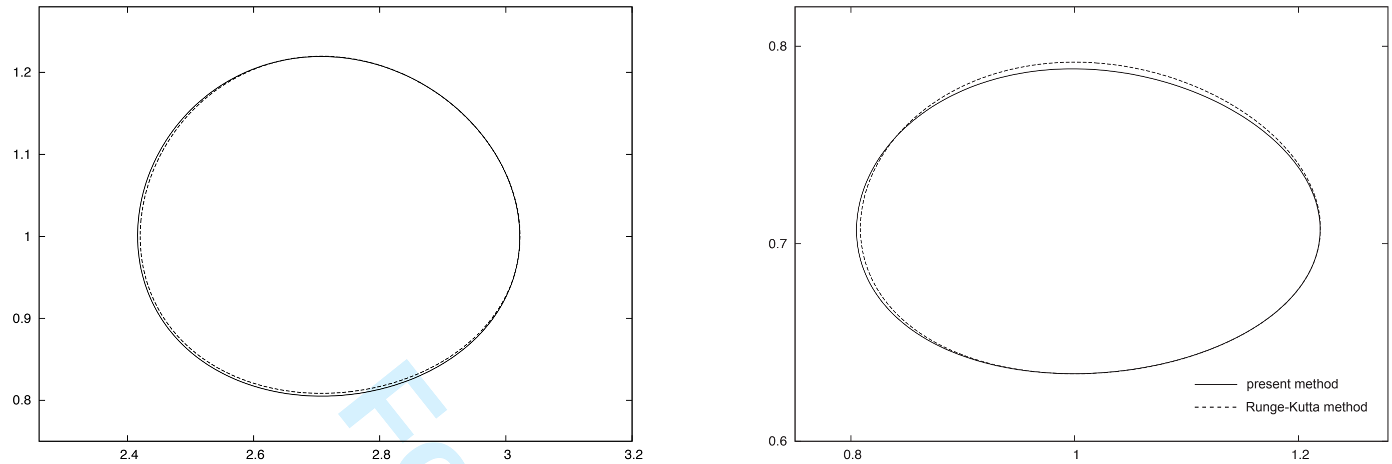

Figure 3. Comparison between the approximate solution to equation (17) with initial conditions $x(0) = 2.864670867$, $y(0) = 1.187254264$, $z(0) = 0.669094392$, obtained by using a fourth order Runge–Kutta method and a fifth order approximation to the solution computed through the Poincaré–Lindstedt method, projected in the $x - y$ (left panel) and $y - z$ (right panel) planes.

6. Conclusion

In this paper, a symbolic algorithm for a general application of the Poincaré–Lindstedt method for the computation of periodic solutions in three–dimensional differential systems of first order has been presented. We have adapted the standard method to three–dimensional nonlinear systems, giving the sufficient conditions to avoid the occurrence of secular terms in the perturbation series solution. The algorithm has been used to compute periodic solutions in a three–dimensional Lotka–Volterra system modeling a chain food interaction.

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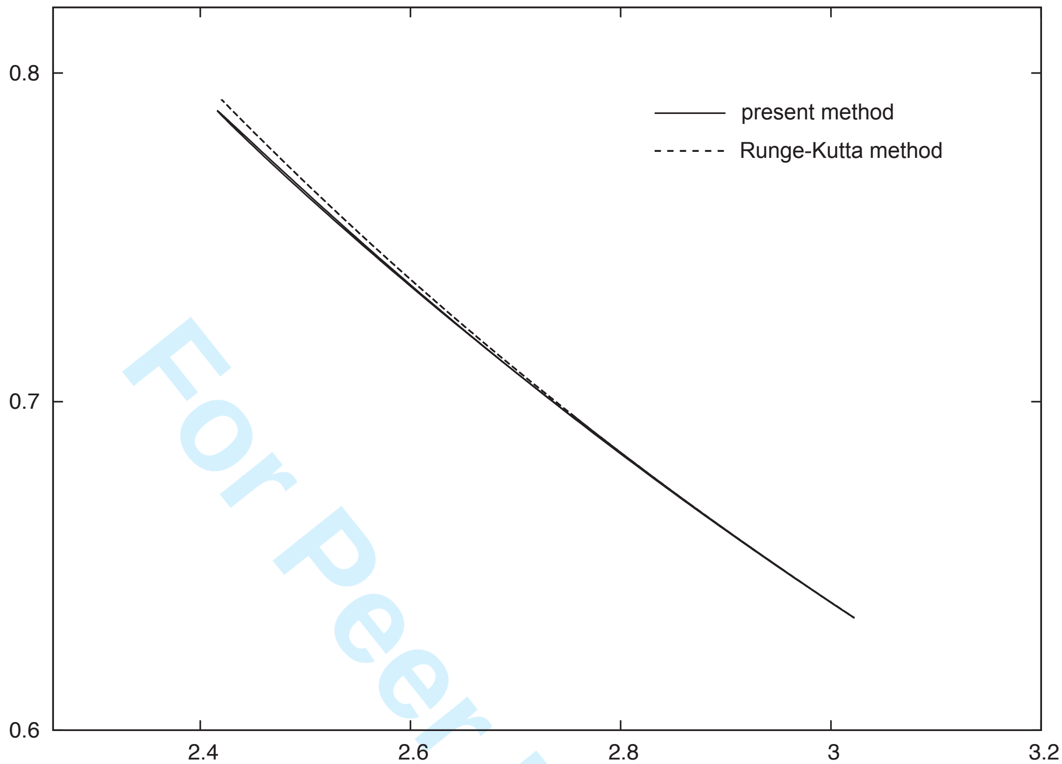

Figure 4. Comparison between the approximate solution to equation (17) with initial conditions $x(0) = 2.864670867$, $y(0) = 1.187254264$, $z(0) = 0.669094392$, obtained by using a fourth order Runge–Kutta method and a fifth order approximation to the solution computed through the Poincaré–Lindstedt method, projected in the $x - z$ plane.

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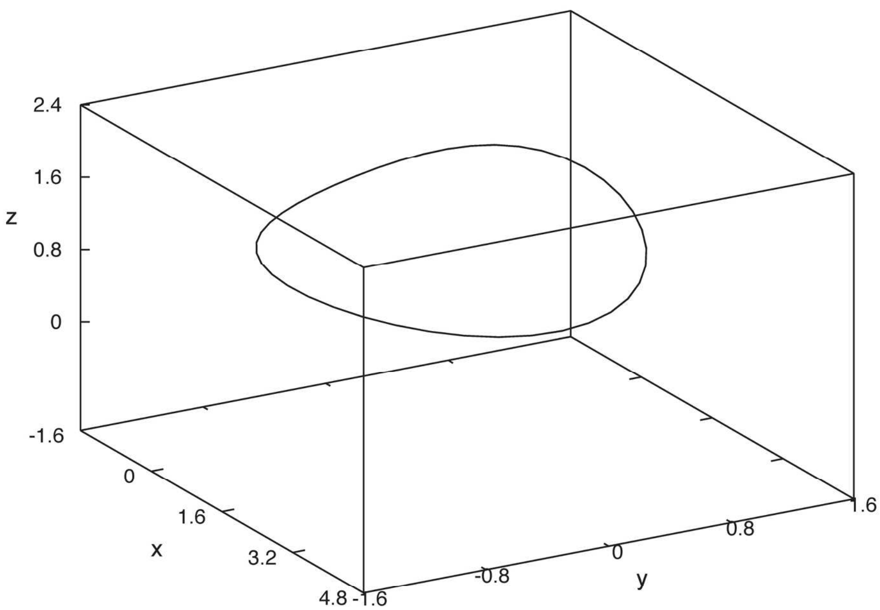

Second order of approximation to the periodic solution to (\ref{ex-1}) with initial conditions $x(0)=2$, $y(0)=1$ and $z(0)=1$.

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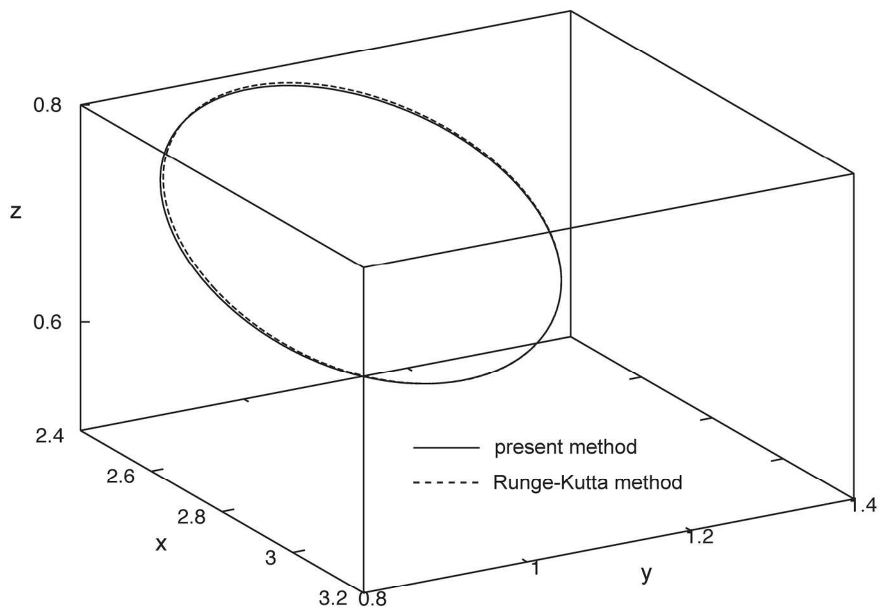
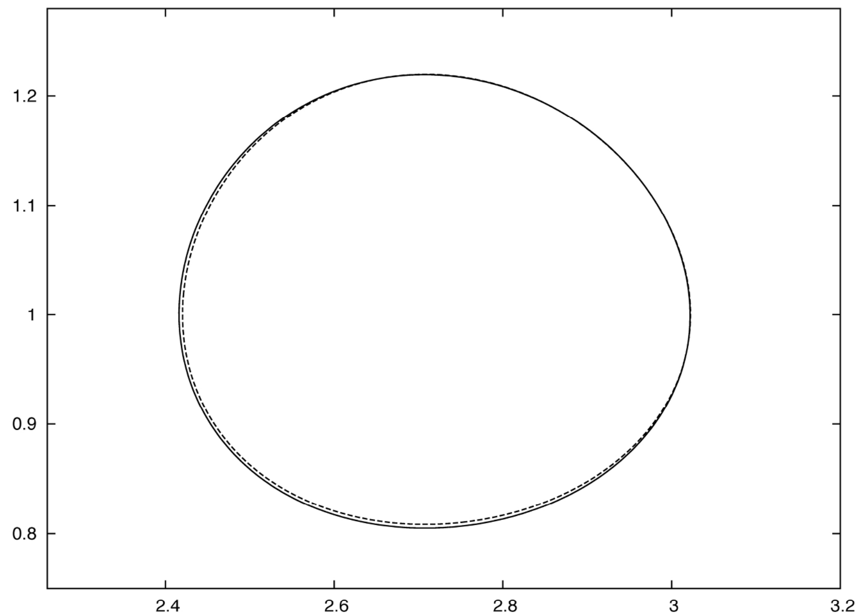


Comparison between the approximate solution to equation (\ref{ejemplito})
with initial conditions
$x(0)=2.864670867$,
$y(0)=1.187254264$,
$z(0)=0.669094392$,
obtained by using a fourth order Runge--Kutta method
and a fifth order approximation to
the solution computed through the Poincar\'e--Lindstedt method.

180x135mm (300 x 300 DPI)



Comparison between the approximate solution to equation (\ref{ejemplito})
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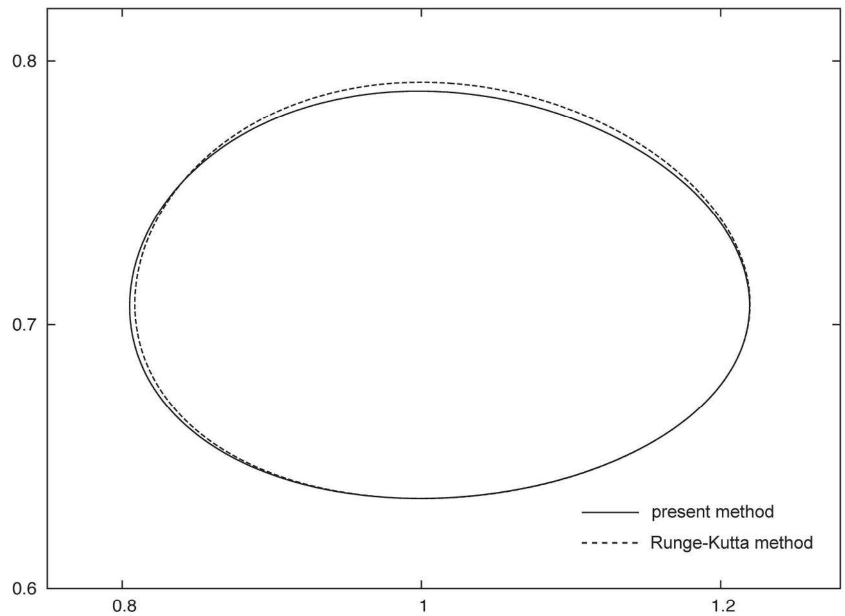
and a fifth order approximation to
the solution computed through the Poincar\'e--Lindstedt method,
projected in the $x-y$ (left panel) and $y-z$ (right panel) planes.

215x166mm (300 x 300 DPI)



Comparison between the approximate solution to equation (\ref{ejemplito})
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$x(0)=2.864670867$,
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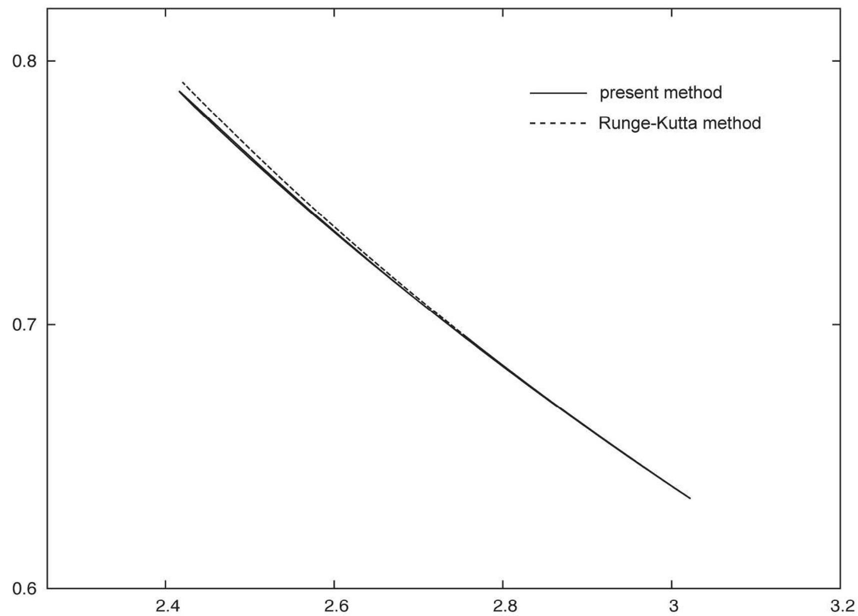
215x166mm (300 x 300 DPI)



Comparison between the approximate solution to equation (\ref{ejemplito}) with initial conditions $x(0)=2.864670867$, $y(0)=1.187254264$, $z(0)=0.669094392$, obtained by using a fourth order Runge--Kutta method and a fifth order approximation to the solution computed through the Poincar\'e--Lindstedt method, projected in the $x-z$ plane.

215x166mm (300 x 300 DPI)